Stochastic Models for Phase Noise

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***Abstract*— We study the mathematical treatment of phase noise in oscillators. The stochastic process is characterized and the relevant quantities for the OFDM performance are discussed. Comparing with line core measurements, we argue that more attention should be paid to the *slow part* of the phase noise.**

I. Introduction

OSCILLATOR phase noise is often a critical item in the receiver design for an OFDM system. There exists a long list of papers that deal with this item. In this connection, the main interest seems to be the *inter-carrier interference* (ICI), see e.g. [1], [2], [3], [4], [5]. It is caused by the *rapid part* of the phase fluctuations and depends on the line wings (the outer parts of the spectral shape).

However, the *slow part* of of the phase fluctuations may cause additional degradations for each single carrier of an OFDM system in a time-variant fading (mobile) radio channel. This is because the OFDM symbol length is typically choosen to be as long as possible to cope with long echoes. The limitation is given by the time variance of the channel. If the time variance of the local oscillator is of the same order, both must be considered together. Measurements suggest that there is really a need for such investigations. To investigate these effects analytically or by computer simulations, one has to choose an appropriate stochastic model for the random signal of the oscillator phase.

We note that this stochastic process is not uniquely determined by the line shape of the oscillator how it can be seen on the spectrum analyzer. To get more insight into its statistical properties, the phase itself must be measured.

In this paper, we discuss stochastic models for this random phase that can be applied to analyse the impact of phase noise on the performance of an OFDM system. We compare with measurements and sketch the direction of future work.

II. Characterization of Random Signals

A. The Oscillator

We consider an oscillator signal of average power equal to one given by

$$x(t) = \sqrt{2}\cos\left(2\pi f_0 t + \varphi(t)\right). \tag{1}$$

The center frequency is denoted by f_0, and $\varphi(t)$ is a random time-variant phase that we model as a stochastic process. Let $z(t) = \exp\left(j\varphi(t)\right)$ be the complex baseband signal corresponding to $x(t)$, i.e.

$$x(t) = \sqrt{2}\,\Re\{z(t)\exp(j2\pi f_0 t)\}. \tag{2}$$

We define the time derivative of the phase, $\omega(t) = \dot{\varphi}(t)$, as the (angular) instantanuous frequency of the signal.

To distinguish between a stochastic process and one of its possible realisations called *sample path*, we use capital letters $X(t)$, $\Phi(t)$, $\Omega(t)$ and $Z(t)$ for the stochastic processes with the sample paths $x(t)$, $\varphi(t)$, $\omega(t)$, and $z(t)$.

B. Autocorrelation and Power Density Spectrum

We assume that the instantanuous angular frequency $\Omega(t) = \dot{\Phi}(t)$ is a stationary process with autocorrelation function (ACF)

$$R_\Omega\left(t\right) = \mathrm{E}\left\{\Omega(t_1+t)\Omega(t_1)\right\}. \tag{3}$$

This implies that $X(t)$ is wide-sense stationary (WSS) [6]. Its ACF, $R_X\left(t\right)$, is related to the complex baseband ACF,

$$R_Z\left(t\right) = \mathrm{E}\left\{Z\left(t_1+t\right)Z^*\left(t_1\right)\right\}, \tag{4}$$

by

$$R_X\left(t\right) = \Re\left\{R_Z\left(t\right)\exp\left(j2\pi f_0 t\right)\right\}. \tag{5}$$

Note that $R_X(0) = R_Z(0) = 1$ is the average total power of the random signal. For a WSS process with ACF $R_X\left(t\right)$, the power spectral density (PSD) $S\left(f\right)$ is given as the Fourier transform of the ACF, i.e.

$$S\left(f\right) = \int_{-\infty}^{\infty} R(t)e^{-j2\pi f t}\mathrm{d}t \tag{6}$$

We denote the ACF for the random phase $\Phi(t)$ as

$$R_\Phi\left(t_1,t_2\right)=\mathrm{E}\left\{\Phi\left(t_1\right)\Phi\left(t_2\right)\right\}. \qquad (7)$$

For the process $\Phi(t)$, the WSS property can often not be assumed. There exist reasonable model processes for $\Phi(t)$ that are not WSS. The most popular one is the Wiener process that we treat in the next section. For such processes, the PSD can not defined by Eq. (6), even though one may attempt to calculate a PSD from a sample path with numerical methods .

C. Relations Between the Spectra

We derive a simple relation between the PSD of a stochastic process and the PSD of its derivative.

Lemma 1: Let $Z(t)$ be a WSS stochastic process (complex or real) with the ACF $R_Z(t)$. Then the time-derivative $\dot{Z}(t)$ is also WSS and its ACF is given by

$$R_{\dot{Z}}(t)=-\ddot{R}_Z(t), \qquad (8)$$

where $\ddot{R}(t)$ denotes the second time-derivative of $R(t)$.

Proof: The ACF of $\dot{Z}(t)$ is given by

$$\mathrm{E}\left\{\dot{Z}\left(t_1\right)\dot{Z}^*\left(t_2\right)\right\}=\frac{\partial^2}{\partial t_1\partial t_2}\mathrm{E}\left\{Z\left(t_1\right)Z^*\left(t_2\right)\right\}. \qquad (9)$$

Using the WSS property of $Z(t)$ we may write

$$\frac{\partial^2}{\partial t_1\partial t_2}R_Z\left(t_1-t_2\right)=-\ddot{R}_Z\left(t_1-t_2\right) \qquad (10)$$

which completes the proof. ■

Corollary 1: If $\Phi(t)$ is WSS, then the relation

$$S_\Phi(f)=\frac{1}{(2\pi f)^2}S_\Omega(f) \qquad (11)$$

holds.

Proof: This follows from the Lemma by setting

$$R_\Omega\left(t\right)=-\ddot{R}_\Phi\left(t\right). \qquad (12)$$

■

To relate the aymptotic behavior of $S_\Omega(f)$ and $S_Z(f)$ for $f\to\pm\infty$, we make the so-called *small angle approximation* (SMAP, see e.g. [2], [4]). We study the behavior for small t which corresponds to small (phase) angles. From

$$\dot{Z}(t)=j\Omega(t)Z(t) \qquad (13)$$

we conclude

$$R_{\dot{Z}}\left(t\right)=\mathrm{E}\left\{\Omega\left(t\right)\Omega\left(0\right)\exp\left(j\int_0^t\Omega\left(t'\right)\mathrm{d}t'\right)\right\}. \qquad (14)$$

For a small phase angle, the exponential can be approximated by one and we may write $R_{\dot{Z}}\left(t\right)\approx R_\Omega\left(t\right)$. From the Lemma we then conclude that for small values of t the relation

$$\ddot{R}_Z(t)\approx-R_\Omega\left(t\right) \qquad (15)$$

holds. From this relation, we may heuristically argue that

$$S_Z(f)\approx\frac{1}{(2\pi f)^2}S_\Omega(f)\quad(f\to\pm\infty) \qquad (16)$$

holds. Note that for this version of the SMAP, it was not necessarry to assume that $\Phi(t)$ is WSS and S_Φ is well-defined. If it is even WSS, we may conclude from (11) and (16) that the relation

$$S_Z(f)\approx S_\Phi(f)\quad(f\to\pm\infty) \qquad (17)$$

holds.

III. Wiener Phase Noise

The Wiener (or Wiener-Lévy) process $W(t)$ was originally introduced as a statistical model to describe certain diffusion processes like Brownian motion [7]. For us, it is most convenient to define $W(t)$ as integrated white noise, i.e.

$$W(t)=\int_0^t N\left(\tau\right)\mathrm{d}\tau, \qquad (18)$$

where $N\left(t\right)$ is white Gaussian noise with an ACF given by

$$R_N(t)=2D_0\delta(t), \qquad (19)$$

where D_0 is called the *diffusion constant*. Because such a Gaussian *phase diffusion* is very easy to analyse, several authors (see e.g. [1], [3], [5]) used the model $\Omega(t)=N(t)$, i.e.

$$\Phi(t)=W(t)+\Phi_0, \qquad (20)$$

where $\Phi_0=\Phi(0)$ is a uniformly distributed (initial) random phase.

The ACF of $W(t)$ can easily be derived as

$$R_W(t_1,t_2)=2D_0\min(t_1,t_2) \qquad (21)$$

for $t_1,t_2\geq 0$. Obviously, $W(t)$ is not WSS, but it is Gaussian with variance

$$\sigma^2=R_W(t,t)=2D_0\left|t\right| \qquad (22)$$

(which reflects Einstein's [8] famous result about Brownian Motion). The ACF $R_Z(t)$ can easily be obtained from the characteristic function $C(k)=\exp\left(-\frac{1}{2}\sigma^2k^2\right)$ for the Gaussian random variable $W(t)$ to be

$$R_Z(t)=\exp(-D_0\left|t\right|). \qquad (23)$$

The corresponding PSD is the Lorentzian

$$S_Z(f) = \frac{2}{D_0}\frac{1}{1+(2\pi f/D_0)^2} \tag{24}$$

with 3dB bandwidth

$$\beta = \frac{D_0}{2\pi}. \tag{25}$$

The PSD $S_\Omega(f) = 2D_0$ is simply white noise. The PSD $S_\Phi(f)$ of the phase $\Phi(t)$ itself is not defined because $\Phi(t)$ is not WSS.

We note that this phase diffusion model is an appropriate model rather for a laser than for a quartz oscillator (see the discussion in [9], [2]). The phase noise sample path of a quartz oscillator (see Figure 1 below) looks very different from Wiener Brownian motion which has a sample path that is continuous, but nowhere differentiable. The reason for using this process it is more or less its simplicity for theoretical analysis and for simulation. Note that the Lorentzian decays very poorly as $\sim f^{-2}$ which may lead to an overestimation of the intercarrier-interference (ICI) for OFDM systems.

IV. General Gaussian Phase Noise

We may now generalize from the Wiener phase noise to arbitrary Gaussian processes. Gaussian processes are also convenient for the theoretical analysis as well as for computer simulations. Note that any Gaussian process can be obtained as suitably filtered white Gaussian noise.

We consider the case that the phase is driven by a general (non-white) mean-zero stationary Gaussian process $\Omega(t)$, i.e.

$$\Phi(t) = \int_0^t \Omega(\tau)\,\mathrm{d}\tau + \Phi_0. \tag{26}$$

A (mean-zero) Gaussian process $\Omega(t)$ can be characterized by the fact that its detector output

$$\Omega_g = \int_{-\infty}^{\infty} g(t)\Omega(t)\mathrm{d}t \tag{27}$$

for any linear measurement given by $g(t)$ is a Gaussian (mean-zero) random variable. The characteristic function of Ω_g is given by

$$C_{\Omega_g}(k) = \mathrm{E}\left\{\exp\left(jk\Omega_g\right)\right\} = \exp\left(-\frac{1}{2}\sigma_g^2 k^2\right) \tag{28}$$

with

$$\sigma_g^2 = \mathrm{E}\left\{\Omega_g^2\right\}. \tag{29}$$

Using stationarity, we find the expression

$$\sigma_g^2 = \int_{-\infty}^{\infty} g(t)\left(R_\Omega * g\right)(t)\,\mathrm{d}t \tag{30}$$

and (by using Parseval's equation)

$$\sigma_g^2 = \int_{-\infty}^{\infty} |G(f)|^2 S_\Omega(f)\,\mathrm{d}f, \tag{31}$$

where $G(f)$ is the Fourier transform of $g(t)$.

We note that for $t \geq 0$, the ACF $R_Z(t)$ is just the characteristic function (28) of Ω_g for $g(\tau) = \mathrm{rect}(\tau/t)$ evaluated at $k=1$. We write

$$R_Z(t) = \mathrm{E}\left\{\exp\left(-\frac{1}{2}\sigma_{\mathrm{rect}}^2(t)\right)\right\} \tag{32}$$

where the expressions

$$\sigma_{\mathrm{rect}}^2(t) = t^2\int_{-\infty}^{\infty} \mathrm{sinc}^2(ft)\, S_\Omega(f)\,\mathrm{d}f \tag{33}$$

and

$$\sigma_{\mathrm{rect}}^2(t) = \int_{-t}^{t} (t-|\tau|)\, R_\Omega(\tau)\,\mathrm{d}\tau \tag{34}$$

can be derived from from Eq. (31).

A. Asymptotic Behavior

Since $t\cdot\mathrm{sinc}^2(ft) \to \delta(f)$ for $t\to\infty$ we obtain

$$\lim_{t\to\infty}\frac{\sigma_{\mathrm{rect}}^2(t)}{t} = S_\Omega(0) = \int_{-\infty}^{\infty} R_\Omega(\tau)\,\mathrm{d}\tau =: 2\alpha \tag{35}$$

which means that

$$R_Z(t) \sim \exp(-\alpha t) \tag{36}$$

for $t\to\infty$. If there is a *correlation time* τ_c of the process $\Omega(t)$ with the property that $R_\Omega(t) \approx 0$ for $t \gg \tau_c$ we conclude from Eqs. (34,35) that $\sigma_{\mathrm{rect}}^2(t) \approx 2\alpha t$ and, thus, the relation (36) holds for $t \gg \tau_c$. Thus, if the correlation time τ_c is small enough compared to the time contant α^{-1} of the exponential decay, i.e. if

$$S_\Omega(0)\tau_c \ll 1, \tag{37}$$

the ACF is governed by (36). Then the line core of $S_Z(f)$ is Lorentzian shaped with $D_0 = \alpha$ in Equation (24).

For the line wings, we conclude from (16) that of $S_Z(f)$ must decay faster than f^{-3} if we can assume that $S_\Omega(f)$ is integrable.

B. Gaussian Processes and Cumulants

We would like to point out that Gaussian processes can be interpreted as the second order approximation of the cumulant expansion of a more general stochastic process, see [10]. The higher-order cumulants are small if

$$\gamma\tau_c \ll 1, \tag{38}$$

where τ_c is the correlation time and γ is the strength of the process that may be defined e.g. by

$$\gamma^2 = \mathrm{E}\left\{\Omega^2(t)\right\}. \tag{39}$$

V. The Effect Of Phase Noise On OFDM

We now consider an OFDM system with carrier spacing $1/T$. If the transmision is corrupted by nothing else but the multiplicative phase noise process $Z(t)$, the Fourier analysis detector output at subcarrier frequency $f_k = k/T$ is given by

$$R_k = \sum_{k'} S_{k'} \Theta_{k'-k} \, , \tag{40}$$

where the S_k are the (PSK or QAM) modulation symbols, and the random variables Θ_k are given by

$$\Theta_k = \frac{1}{T} \int_0^T \exp\left(j2\pi kt/T\right) Z\left(t\right) \mathrm{d}t. \tag{41}$$

A. *Inter-Carrier Interference (ICI)*

The terms in the sum (40) with $k' \neq k$ are undesired contributions (due to the loss of orthogonality) from other subcarriers and are thus named inter-carrier interference (ICI) terms. They are due to the rapid part of the phase fluctuations. The contributions are statistically independent if the factors $S_{k'}$ can assumed to be independent zero-mean random variables. The variance of each of them can be calculated, and the *signal-to-interference ratio* can be obtained as [4]

$$SIR_k = \left(\int_{-\infty}^{\infty} S_Z\left(f\right) W_{ICI,k}\left(fT\right) \mathrm{d}f \right)^{-1} \tag{42}$$

with the ICI weighting function

$$W_{ICI,k}\left(x\right) = \sum_{k' \neq k} \mathrm{sinc}^2\left(x - \left(k - k'\right)\right) \tag{43}$$

which can conveniently be approximated by a simpler function if necessary. When the line wings of $S_X(f)$ are known by measurement, the SIR can thus easily be evaluated.

B. *Slow Phase Fluctuations*

In the sum in Eq. (40), the term $S_k\Theta_0$ corresponding to $k' = k$ contains the desired information symbol S_k but phase-shifted by a phasor

$$\Theta_0 = \frac{1}{T} \int_0^T Z\left(t\right) \mathrm{d}t \tag{44}$$

that is common to all subcarriers. Its average phase-shift $\bar{\Phi}$ is thus named the *common phase error* (CPE) and can be characterized by

$$\bar{\Phi}^2 = \mathrm{E}\left\{ \Theta_0^2 \right\}. \tag{45}$$

In the literature, the CPE is usually not regarded as a critical item if continuous pilot carriers have been inserted into the OFDM symbols to estimate (and correct) this phase error [4], [2]. A frequency-selective static fading channel does not hurt. In that case, $S_k\Theta_0$ has only to be replaced by $S_k H_k \Theta_0$, where H_k is the complex fading amplitude at the frequency f_k. For a time-variant channel transfer function $H_k(t)$, however, the desired term is given by $S_k \Xi_k$ with

$$\Xi_k = \frac{1}{T} \int_0^T H_k\left(t\right) Z\left(t\right) \mathrm{d}t. \tag{46}$$

Now the time variance is no longer common to all frequencies, and the time-variant fading at each subcarrier frequency f_k is affected by an additional multiplicative factor due to the phase noise. If the system has already to cope with *fast fading*, the problems become more severe if the time variance of the phase-noise is in the same order as time variance of the channel. The Doppler spectrum $S_D(f)$ experiences an additional broadening by $S_Z(t)$. The resulting spectrum is given by the convolution $S_D(f) * S_Z(f)$, and its bandwidth is approximately the sum of the bandwidth of the components. For a system with differential PSK (like DAB), the phase errors of both components add up. For a system with coherent demodulation, the channel estimation will be degraded due to the slow phase fluctuations over several OFDM symbols. It is surprising that very few attention has paid to this problems in the literature.

We note that even in a static Gaussian channel, the CPE becomes more severe if the OFDM carrier spacing $1/T$ decreases. This can easily be understood for Wiener phase noise. From Einstein's equation (22) one may argue that $\bar{\Phi}^2$ grows linearly with T. This can be proven. In fact, we can easily derive from Eqs. (21), 44, and (45) that for Wiener phase noise the equation

$$\bar{\Phi}^2 = \frac{2}{3} D_0 T \tag{47}$$

holds. We expect a similar behavior for other processes.

VI. An Example of Measured Phase Noise

To illustrate the theoretical concepts, we refer to a practical example. Because ICI effects on OFDM systems are already well-understood, we focus our attention to the line core. We study a quartz oscillator stabilized by a PLL and located at the center frequency $f_0 = 24.576$ MHz. After down-conversion to frequency zero, one million samples of the quadrature components were recorded[1] with

[1]The author would like to thank Dr. Frank Hofmann at Robert Bosch GmbH for providing him with these data sampled from a laboratory prototype.

a sampling frequency of $f_s = 24000$ Hz, leading to a measured sample path of 41.667 seconds. A residual frequency offset has been corrected digitally. Figure 1 shows the sample path of the phase obtained from

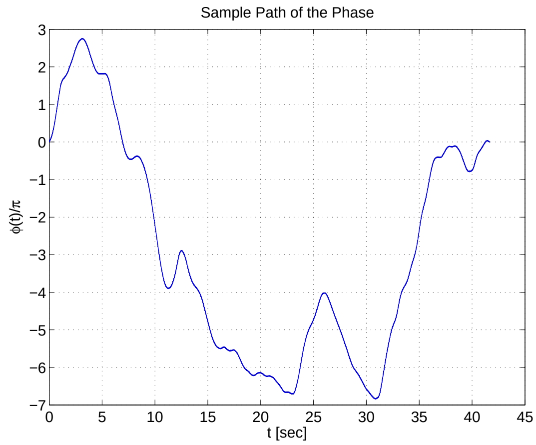

Fig. 1. Sample path of the measured Phase.

this measurement. One can see clearly that there are slow fluctuations in the phase on the time scale of one second, which leads to the conclusion that there is a correlation time τ_c of that order. Figure 2 shows the spectra $S_Z(f)$, $S_\Phi(f)$, and $\frac{1}{(2\pi f)^2} S_\Omega(f)$ on a logarithmic scale. We observe a decay of 40

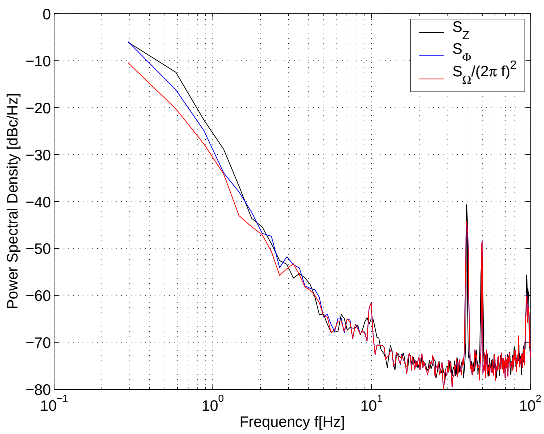

Fig. 2. The three spectra.

dB between 1 Hz and 10 Hz, which means that $S_Z(f) \sim f^{-4}$ in this region. From both figures we conclude that a Wiener process is not a suited model for this oscillator. We note that the noise floor that can be observed is due to the quantization. It is much lower if a high precision spectrum analyser is used. It is interesting to look at the line core which has a width in the order of one Hertz. These low-frequency components may become quite important e.g. in a DRM system where the Doppler spread is of the same order. From a more detailed numerical analysis we found that both conditions (37) and (38) for Wiener and Gaussian processes are not fulfilled and, thus, more sophicticated models are necessary to reflect the reality.

VII. Discussion and Conclusions

We have discussed the parameters that characterize oscillator phase noise to understand their impact on OFDM systems. ICI effects are well understood and have been treated frequently in the literature. These additive perturbations depend only on the wings of the spectral line shape $S_Z(f)$, but not directly on the statistical properties of the phase. For simulations, we may thus use any model process.

In contrast, the effects of the phase error $\bar{\Phi}$ is much less understood and its effect on an OFDM system is often neglected. The measured phase noise example shows that one should take more care of it. To analyse the OFDM system performance by numerical simulations, either a reasonable statistical model must be chosen (Wiener and Gaussian seems to be too simple), or the the recorded sample paths themselves may be used for simulations. This is our starting point for future work.

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